Mathematica Moravica
Vol. **16-2** (2012), 21–24

On ACVF of a Regime Switching AR(1) Process

Reza Habibi

Abstract. This paper considers the auto-covariance function (ACVF) of a regime switching AR(1) process. Two independent Markov chains governs on auto-regressive coefficient and standard deviation of white noise process. Our approach to solve this problem is to obtain the ACVF of a AR(1) model with time varying parameters and then extend this result to regime switching case. Finally, an application of our formulae in model selection is proposed.

1. Introduction and Main Results

Consider a first order zero mean regime switching auto-regressive RS-AR(1) process defined by

$$Y_t = \phi_{S_t^1} Y_{t-1} + \sigma_{S_t^2} Z_t,$$

where $|\phi_r| \leq |\phi| < 1$ and $\sigma_r \leq M$ for all $r = 0, \pm 1, \pm 2, \ldots$. Here, S_t^1 and S_t^2 are two independent Markov chains with the same state space $\mathfrak{S}$. Let Z_t be a white noise process with zero mean and variance σ^2. In this model, we assume that Z_t is independent of Y_{t-1} and S_t^1 and S_t^2 are independent of Y_{t-1} and Z_t. In this paper, we are going to derive an expression for ACVF in this non-stationary process, i.e.,

$$\gamma_t(h) = \operatorname{cov}(Y_{t+h}, Y_t), \quad h = 0, \pm 1, \pm 2, \ldots$$

The regime switching model deals with the capturing structural changes in the underlying financial time series using the time ordered observations. An example is Cryer and Chan (2008) who applied this type of time series for modeling the stock returns. In this model, the parameters such as mean and/or volatility vary through the sample, in fact they are functions of some Markov chain processes. This models has been applied in regression analysis, Box-Jenkins time series and as well as in GARCH modeling of financial problems. An excellent reference in this field is Zivot and Wang (2006).

2010 *Mathematics Subject Classification.* Primary 62G20; 62F20.

Key words and phrases. Auto-covariance function; Auto-regressive model; Markov chain; Non-stationary process; Regime switching.

To derive $\gamma_t(h)$, first consider a zero mean AR(1) process Y_t^* with time varying parameters ϕ_t^* and σ_t^* defined by

$$Y_t^* = \phi_t^* Y_{t-1}^* + \sigma_t^* Z_t^*,$$

where $|\phi_r^*| \leq |\phi^*| < 1$ and $\sigma_r^* \leq M^*$ and $Z_t^* \sim$WN$(0, \sigma^{*2})$. Suppose that $E(Y_t^*) = 0$ and note that

$$v_t^* = \mathrm{var}(Y_t^*) \leq \phi^{*2} v_{t-1}^* + M^* \sigma^{*2}.$$

Therefore, we see that

$$\begin{aligned} v_t^* &\leq M^* \sigma^{*2}(1 + \phi^{*2} + \cdots + \phi^{*2(t-1)}) + \phi^{*2t} v_1^* \\ &= M^* \sigma^{*2} \frac{(1 - \phi^{*2t})}{(1 - \phi^{*2})} + \phi^{*2t} v_1^* \\ &\leq M^* \sigma^{*2} \frac{(1 - \phi^{*2t})}{(1 - \phi^{*2})} + v_1^* = UB. \end{aligned}$$

Using a recursive solution, we find that

$$\begin{aligned} E(Y_t^* - \sum_{i=0}^{k} d_i^{*t} Z_{t-i}^*)^2 &= \prod_{i=0}^{k} \phi_{t-i}^{*2} E(Y_t^{*2}) \\ &\leq |\phi^*|^{2(k+1)} UB \\ &\to 0, \quad \text{as } k \to \infty, \end{aligned}$$

where the coefficients d_i^{*t} are given as follows

$$d_i^{*t} = \sigma_{t-i}^* \prod_{j=1}^{i} \phi_{t-j+1}^*.$$

Therefore, with probability one, we conclude (by letting $k \to \infty$) that

$$Y_t^* = \sum_{i=0}^{\infty} d_i^{*t} Z_{t-i}^*.$$

There is another way to obtain this equation. Let B denote the backward operator, i.e. $BY_t^* = Y_{t-1}^*$. Note that

$$Y_t^* = \frac{1}{1 - \phi_t^* B} (\sigma_t^* Z_t^*),$$

which equals to (using Taylor expansion)

$$Y_t^* = \sum_{i=0}^{\infty} \prod_{j=1}^{i} \phi_{t-j+1}^* B^i (\sigma_t^* Z_t^*) = \sum_{i=0}^{\infty} d_i^{*t} Z_{t-i}^*.$$

Using the above equation, it is not difficult to see that

$$\mathrm{cov}(Y_t^*, Y_{t+h}^*) = \sigma^{*2} \sum_{i=0}^{\infty} d_i^{*t} d_{i+h}^{*t}.$$

Now, we are on position to calculate the $\gamma_t(h)$. Define the σ-field $\mathcal{F}$ constructed using the whole information of two Markov chains up to time t as follows

$$\mathcal{F} = \sigma(\{S^1_{t-i}\}_{i=0}^{\infty}, \{S^2_{t-i}\}_{i=0}^{\infty})$$

and let $D_i^t = \sigma(S^2_{t-i}) \prod_{j=1}^{i} \phi(S^1_{t-j+1})$. It is easy to see that

$$E(Y_t Y_{t+h}|\mathcal{F}) = \sigma^2 \sum_{i=0}^{\infty} D_i^t D_{i+h}^t.$$

Therefore, using total probability law, we understand that

$$\gamma_t(h) = \sigma^2 \sum_{i=0}^{\infty} E(D_i^t D_{i+h}^t).$$

However, it seems that calculating the expectation $E(D_i^t D_{i+h}^t)$ be hard in practice. As follows, we give a technique which simplifies calculating this mean. To this end, suppose that U_t, $t \geq 0$ is a Markov chain with a finite state space. To calculate $E(\prod_{j=1}^{p} U_j)$, we first suppose that $p = 2$, then $E(U_1U_2) = E(U_1E(\ U_2|U1))$. Next, let $p = 3$, then

$$\begin{aligned} E(U_1U_2U_3) &= E(U_1U_2E(U_3|U_1, U_2)) \\ &= E(U_1U_2E(U_3|U_2)) : \text{Markov property} \\ &= E(U_1E(U_2|U_1)E(U_3|U_2)). \end{aligned}$$

Since the conditional distributions exist then this expectations can be calculated. For other choices of p the same method is applied. This technique is applicable for $E(\prod_{j=1}^{p} h_j(U_j))$ for some measurable functions h_j, $j = 1, 2, \ldots, p$, for more description on this method, see Iacus (2008). Note that, in practice, we can estimate $E(D_i^t D_{i+h}^t)$ using a Monte Carlo technique with adding some variance reduction methods, see Brazzale *et al.* (2007).

2. Application

In this section, we use an application of formulae obtained in the previous section. As we see, the ACVF is a function of expectation of some product of Markov chains. As, it is seen this expectation decays exponentially, as $h \to \infty$. Therefore, this property is transferred to ACVF. Up to this time, $\gamma_t(h)$ behaves like the ACVF of a ordinary AR(1). However, the rate of decay to zero is different form time point t to $t + 1$. Therefore, an unified strategy for selecting a RS-AR(1) is to monitor the time series plot of a series and if it seems there are some changes in its level, mean or variance, and if there is an exponential-wise decay to zero with different rates, then select an RS-AR model and therefore, estimate the related model parameters and do suitable statistical inference.

REFERENCES

[1] A.R. Brazzale, A.C. Davison and N. Reid, *Applied Asymptotic Case Studies in Small-Sample Statistics*, Cambridge University Press, 2007.

[2] J.D. Cryer and K.S. Chan, *Time Series Analysis: With Applications in R*, 2nd Edition, Springer Science + Business Media, New York, 2008.

[3] S.M. Iacus, *Simulation and Inference for Stochastic Differential Equations: With R Examples*, Springer Science + Business Media, New York, 2008.

[4] E. Zivot and J. Wang, *Modeling Financial Time Series with S-PLUS*, Second edition, Springer Science + Business Media, New York, 2006.

REZA HABIBI
DEPARTMENT OF STATISTICS
CENTRAL BANK OF IRAN
IRAN
E-mail address: habibi1356@gmail.com